October Math Gems

Problem of the week 20

§1 problems

Problem 1.1. Find all real numbers x, y, z so that

$$x^2y + y^2z + z^2 = 0$$
$$z^3 + z^2y + zy^3 + x^2y = \frac{1}{4}(x^4 + y^4).$$

answer. Notice we can derive two relations from the first equation, which are:
$(I) : (z + \frac{y^2}{2})^2 = \frac{y^4}{4} - x^2y$
$(II) : (\frac{x^2}{2} + yz)^2 = \frac{x^4}{4} - z^3$
Summing these two, we get: $(z+\frac{y^2}{2})^2+(\frac{x^2}{2}+yz)^2 = \frac{x^4+y^4}{4}-z^3-x^2y = z^2y+zy^3 = -(xy)^2$
Hence, $(z+\frac{y^2}{2})^2+(\frac{x^2}{2}+yz)^2+(xy)^2 = 0$ and now is just conclude that $x = y = z = 0$ $\square$

Problem 1.2. If x is a real number satisfying the equation

$$9\log_3 x - 10\log_9 x = 18\log_{27} 45,$$

then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

answer. We have $\frac{9\log x}{\log 3} - \frac{5\log x}{\log 3} = \frac{6\log 45}{\log 3}$, so $4\log x = 6\log 45$.
Removing the logs we have $x^4 = 45^6$, or $x = 45\sqrt{45} \implies 135\sqrt{5} \implies 135+5 = \boxed{140}$ $\square$

Problem 1.3. Find all primes p, such that there exist positive integers x, y which satisfy

$$\begin{cases} p + 49 = 2x^2 \\ p^2 + 49 = 2y^2 \end{cases}$$

answer. Even removing the requirement that p is prime, this is an elliptic curve and it is possible to determine all integral points on it.
More precisely, we can rewrite it as $(2x^2 - 49)^2 = 2y^2 - 49$. Multiplying by $2x^2$ gives us

$$Y^2 = X^3 - 98X^2 + 2450X$$

where $X = 2x^2$ and $Y = 2xy$.
A computer algebra system can be used to compute the Mordell-Weil rank, which is equal to 2, and to determine all the integral points (up to sign and ignoring the infinity point):

$$(X, Y) = (0, 0), (25, 125), (32, 104), (49, 49), (50, 50), (72, 204), (98, 490), (9800, 965300).$$

Translating back into (x, y, p), we find the following solutions:

$$(x, y, p) = (0, 35, -49), (4, 13, -17), (5, 5, 1), (6, 17, 23), (7, 35, 49), (70, 6895, 9751).$$

Among these, only $(6, 17, 23)$ gives prime value of p. $\square$

Problem 1.4. Suppose x and y are real numbers satisfying

$$\begin{cases} x^3 - y^3 = 493. \\ x^2y - y^2x = 50. \end{cases}$$

What is the positive difference between x and y?

answer.

$$(x-y)^3 = x^3 - 3x^2y + 3y^2x - y^3$$
$$(x-y)^3 = (x^3 - y^3) - 3(x^2y - y^2x)$$
$$(x-y)^3 = 493 - 150 = 343$$
$$x - y = 7$$

□

Problem 1.5. Find all pairs (x, y) of real numbers satisfying the system : $\begin{cases} x + y = 2 \\ x^4 - y^4 = 5x - 3y \end{cases}$

answer. From the first equation, $x + y = 2 \implies y = 2 - x$. Substituting this to the second equation gives

$$x^4 - (2-x)^4 = 5x - 3(2-x)$$
$$8x^3 - 24x^2 + 24x - 10 = 0$$
$$8(x-3)^3 - 2 = 0$$

This shows us that $x = 3 + \frac{1}{\sqrt[3]{4}}$

The only pair is $\boxed{\left(3 + \frac{1}{\sqrt[3]{4}}, -1 - \frac{1}{\sqrt[3]{4}}\right)}$. □

Problem 1.6. Solve the equation in $\mathbb{R}$, the system $\begin{cases} x + y + xy = 4 \\ y + z + yz = 7 \\ x + z + xz = 9 \end{cases}$

answer. $(x+1)(y+1) = 5$ $(y+1)(z+1) = 8$ $(x+1)(z+1) = 10$
Multiplying this all together, $(x+1)^2(y+1)^2(z+1)^2 = 400 \implies (x+1)(y+1)(z+1) = \pm 20$.
Dividing by the original equations, $z + 1 = \pm 8$, $x + 1 = \pm\frac{5}{2}$, $y + 1 = \pm 2$.
This leads to the solutions $(\frac{3}{2}, 1, 7)$ and $(\frac{-7}{2}, -3, -9)$. □

Problem 1.7. If a dan b are positive numbers and satisfy, $^a log 4 =^b log 10 =^{a-b} log 25$
What are the value of a and b?

answer. Given $\log_a 4 = \log_b 10 = \log_{a-b} 25$. Then $\log_4 a = \log_{10} b = \log_{25}(a-b) = t \Rightarrow$ $4^t = a$ and $10^t = b$ and $25^t = a - b$, $2^t = \sqrt{a}$ and $10^t = b$ and $5^t = \sqrt{a-b}$. $\sqrt{a} \cdot \sqrt{a-b} = b$. $a^2 - ab - b^2 = 0$. $a = b \cdot \frac{\sqrt{5}+1}{2}$ (1). From $\log_4 a = \log_{10} b$, we have $\frac{\ln a}{\ln 4} = \frac{\ln b}{\ln 10}$ (2). □

Problem 1.8. Solve

$$\log_x \left(\frac{x^{4x-6}}{2}\right) = 2x - 3.$$

answer. $4x - 6 - \log_x 2 = 2x - 3$ $2x - 3 = \frac{1}{\log_2 x}$ $(2x-3)\log_2 x = 1$ $x = 2$ □

Problem 1.9. Let a, b, and c be distinct positive integers such that $\sqrt{a} + \sqrt{b} = \sqrt{c}$ and c is not a perfect square. What is the least possible value of $a + b + c$?

answer. Squaring gives $a + b + 2\sqrt{ab} = c$, so ab must be a perfect square.
Obviously $ab = 1$ doesn't work, and neither does $ab = 4, 9$.
If $ab = 16$, then $a = 2, b = 8$ gives $c = 18$ and an answer of $\boxed{28}$. □

Problem 1.10. Solve this system of equations

$$\begin{cases} x^2 = y^3 + 1 \\ y^2 = x^3 - 23 \end{cases}$$

answer.

$$x^2 = y^3 + 1 \implies x = \pm\sqrt{y^3 + 1}$$

Now, plugging $x = \sqrt{y^3 + 1}$ in $y^2 = x^3 - 23$

$$y^2 = (\sqrt{y^3 + 1})^3 - 23 \implies y = 2$$

$$y^2 = (y^3 + 1)^{\frac{3}{2}} - 23 \implies (y^2 + 23)^2 = (y^3 + 1)^3$$

$$y^4 + 46y^2 + 529 = (y^3 + 1)^3$$

$$(y - 2)(y^8 + 2y^7 + 4y^6 + 11y^5 + 22y^4 + 43y^3 + 89y^2 + 132y + 264) = 0$$

$$\begin{cases} y - 2 = 0 \implies y = 2 \\ (y^8 + 2y^7 + 4y^6 + 11y^5 + 22y^4 + 43y^3 + 89y^2 + 132y + 264) = 0 \qquad \text{has no solution} \in R \end{cases}$$

For $x = -\sqrt{y^3 + 1}$ in $y^2 = x^3 - 23$, there is no solution $\in R$.
Now, for getting x substitute with y in one of these equations:

$$x^2 = 8 + 1 \implies x = 3$$

$$x = 3 \qquad y = 2$$

□

Problem 1.11. Starting with a 5×5 grid, choose a 4×4 square in it. Then, choose a 3×3 square in the 4×4 square, and a 2×2 square in the 3×3 square, and a 1×1 square in the 2×2 square. Assuming all squares chosen are made of unit squares inside the grid. In how many ways can the squares be chosen so that the final 1×1 square is the center of the original 5×5 grid?

answer. It doesn't matter which 4×4 square you choose. WLOG let's assume you chose the bottom left 4×4 square. Then you have 4 cases. Top right 3×3 square Top left 3×3 square Bottom right 3×3 square Bottom left 3×3 square Thus, there are $4 + 2 + 2 + 1 = 9$ ways. Since there are 4 possible 4×4 squares our answer is $\boxed{36}$ □

Problem 1.12. Let $ABCD$ be a rectangle with $AB = 10$ and $AD = 5$. Suppose points P and Q are on segments CD and BC, respectively, such that the following conditions hold: $BD \parallel PQ$ $\angle APQ = 90°$. What is the area of $\triangle CPQ$?

answer. Slope of PQ is $\frac{1}{2}$ thus, the slope of AP is -2. Therefore $PC = \frac{15}{2}$ and $CQ = \frac{15}{4}$ so our answer is $\boxed{\dfrac{225}{16}}$ □

Problem 1.13. How many real roots does this log equation have?

$$\log_{(x^2-3x)^3} 4 = \frac{2}{3}$$

Should I use the fundamental theorem of algebra for this problem?

answer. $\log_{(x^2-3x)^3} 4 = \frac{2}{3} \Leftrightarrow \frac{2}{3}\log_{(x^2-3x)} 2 = \frac{2}{3} \Leftrightarrow \log_{(x^2-3x)} 2 = 1 \Leftrightarrow x^2 - 3x = 2 \Leftrightarrow x^2 - 3x - 2 = 0$ □

Problem 1.14. In trapezoid $ABCD$, leg $\overline{BC}$ is perpendicular to bases $\overline{AB}$ and $\overline{CD}$, and diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular. Given that $AB = \sqrt{11}$ and $AD = \sqrt{1001}$, find BC^2.

answer. Label the intersection of the diagonals E. $AE^2 + BE^2 = 11$, $BE^2 + CE^2 = BC^2$, $AE^2 + DE^2 = 1001$, so we have $11 - BE^2 = 1001 - DE^2 \Rightarrow BE^2 = DE^2 - 990$. Substituting, $CE^2 + DE^2 - 990 = BC^2 = CD^2 - 990$. Drawing the perpendicular AF to DC, we have $CF = \sqrt{11}$. $AF^2 = BC^2 = 1001 - (CD - \sqrt{11})^2 = 990 - CD^2 + 2\sqrt{11} = CD^2 - 990$, so we have $CD^2 - CD\sqrt{11} - 990 = 0 \Rightarrow CD = 10\sqrt{11}$. Therefore, we have $BC^2 = CD^2 - 990 = \boxed{110}$. □

Problem 1.15. Find

$$\cos\frac{2\pi}{2013} + \cos\frac{4\pi}{2013} + \cdots + \cos\frac{2010\pi}{2013} + \cos\frac{2012\pi}{2013}$$

answer. Let $z = e^{i\pi/2013}$ be a 2013^{th} root of unity. We have $\cos(\frac{n\pi}{2013}) = \frac{z^n + z^{2013-n}}{2}$. Thus,

$$\cos(\tfrac{2\pi}{2013}) + \cos(\tfrac{4\pi}{2013}) + \ldots + \cos(\tfrac{2012\pi}{2013}) = \tfrac{1}{2}[z^2 + z^{2013-2} + z^4 + z^{2013-4} + \ldots z^{2012} + z^{2013-2012}]$$

$$= \tfrac{1}{2}[z + z^2 + z^3 + \ldots + z^{2012}]$$

$$= \tfrac{1}{2}[1 + z + z^2 + z^3 + \ldots + z^{2012}] - \tfrac{1}{2}$$

$$= \frac{z^{2013} - 1}{2(z-1)} - \frac{1}{2}$$

$$= \boxed{\frac{-1}{2}}$$

□

Problem 1.16. Find all triples (a, b, c) of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

answer. First assume that none of the numbers is equal to zero. Then considering the first equation we have

$$a(ab-1) = c(b^2-1)$$

By the condition we have $ab - 1 = -c(a+b)$ so dividing by c we get

$$a(a+b) = 1 - b^2 \implies a^2 + ab = 1 - b^2$$

Now add the other two similar relations to find $2(a^2+b^2+c^2) = 3 - ab - bc - ca = 2$, so $a^2+b^2+c^2 = 1$. Then $(a-b)^2+(b-c)^2+(c-a)^2 = 0$ implying $a = b = c = \frac{1}{\sqrt{3}}$, which works.

Now wlog assume $a = 0$, then we get $c = b^2c = b$, so in particular $b = c$ and $ab + bc + ca = b^2 = 1$, giving $b = c = 1$ or $b = c = -1$, and both work.

In summary the solutions are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, $(0, 1, 1)$ and $(0, -1, -1)$, along with cyclic permutations. □

Problem 1.17. Solve over $\mathbb{R}$ the equation $4^{(sinx)^2} + 3^{(tanx)^2} = 4^{(cosx)^2} + 3^{(cotanx)^2}$.

answer. Setting $t = \cos^2 x \in (0,1)$, this is $4^{1-t} + 3^{\frac{1-t}{t}} = 4^t + 3^{\frac{t}{1-t}}$ LHS is decreasing while RHS is increasing and so at most one real root over $(0,1)$ $t = \frac{1}{2}$ is a trivial root and so is the only one. And so $\cos^2 x = \frac{1}{2}$ which is $\boxed{x = \frac{\pi}{4} + k\frac{\pi}{2}}$ □

Problem 1.18. Two distinct, real, infinite geometric series each have a sum of 1 and have the same second term. The third term of one of the series is $1/8$, and the second term of both series can be written in the form $\frac{\sqrt{m}-n}{p}$, where m, n, and p are positive integers and m is not divisible by the square of any prime. Find $100m + 10n + p$.

answer. Let the second term of each series be x. Then, the common ratio is $\frac{1}{8x}$, and the first term is $8x^2$.

So, the sum is

$$\frac{8x^2}{1 - \frac{1}{8x}} = 1$$

. Thus,

$$64x^3 - 8x + 1 = (4x-1)(16x^2+4x-1) = 0 \Rightarrow x = \frac{1}{4}, \frac{-1 \pm \sqrt{5}}{8}$$

.

The only solution in the appropriate form is $x = \frac{\sqrt{5}-1}{8}$. Therefore, $100m + 10n + p = \boxed{518}$. □

Problem 1.19. Let a, b, c, and d be real numbers that satisfy the system of equations

$$\begin{aligned} a + b &= -3 \\ ab + bc + ca &= -4 \\ abc + bcd + cda + dab &= 14 \\ abcd &= 30. \end{aligned}$$

There exist relatively prime positive integers m and n such that

$$a^2 + b^2 + c^2 + d^2 = \frac{m}{n}.$$

Find $m + n$.

answer. Let $d = \frac{30}{abc}$. Equation (3) becomes :

$$abc + \frac{30(ab + bc + ca)}{abc} = abc - \frac{120}{abc} = 14$$

Hence $(abc)^2 - 14(abc) - 120 = 0$. Solving the above we get $abc = -6$ or $abc = 20$. $ab + bc + ca = ab - 3c = -4 \Longrightarrow ab = 3c - 4$. If $abc = -6$, then $c(3c - 4) = -6$. This gives does not give real solutions for c. Hence $abc = 20$. $c(3c - 4) = 20 \implies c = -2$ or $c = \frac{10}{3}$. If $c = 10/3$, then the system $a + b = -3$ and $ab = 6$. Does not have real solutions for both a and b. Hence $c = -2$.

$$a^2 + b^2 + c^2 + d^2 = 9 + \frac{9}{4} - 2ab + 4 = 9 + 20 + 4 + \frac{9}{4} = \frac{141}{4}$$

The answer is $\boxed{145}$. □

Problem 1.20. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has 10 times the sum of the original series. The common ratio of the original series is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

answer. Let's call the first term of the original geometric series a and the common ratio r, so $2005 = a + ar + ar^2 + \ldots$.
Using the sum formula for infinite geometric series, we have $\frac{a}{1-r} = 2005$. Then we form a new series,

$$a^2 + a^2r^2 + a^2r^4 + \ldots$$

We know this series has sum $20050 = \frac{a^2}{1-r^2}$. Dividing this equation by $\frac{a}{1-r}$, we get $10 = \frac{a}{1+r}$. Then $a = 2005 - 2005r$ and $a = 10 + 10r$ so $2005 - 2005r = 10 + 10r$, $1995 = 2015r$ and finally $r = \frac{1995}{2015} = \frac{399}{403}$, so the answer is $399 + 403 = \boxed{802}$. □